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NORMS OF TOEPLITZ MATRICES WITH FISHER-HARTWIG SYMBOLS*

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Abstract. We describe the asymptotics of the spectral norm of finite Toeplitz matrices generated by functions with Fisher-Hartwig singularities as the matrix dimension goes to infinity. In the case of positive generating functions, our result provides the asymptotics of the largest eigenvalue, which is of interest in time series with long-range memory.

Key words. Toeplitz matrix, spectral norm, Fisher-Hartwig singularity, time series, long range memory

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1. Introduction. Let $\{a_k\}_{k\in\mathbf{Z}}$ be a sequence of complex numbers and denote by T_n the $n\times n$ Toeplitz matrix $(a_{j-k})_{j,k=0}^{n-1}$. We are interested in the behavior of the spectral norm $\|T_n\|$ as $n\to\infty$. Notice that if the matrix T_n is positive definite, then $\|T_n\|$ is just the maximal eigenvalue of T_n.

If there is a function $a\in L^1(\mathbf{T})$ such that $\{a_k\}_{k\in\mathbf{Z}}$ is the sequence of the Fourier coefficients of a, that is, $a_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}a(e^{i\theta})e^{-ik\theta}d\theta$, we call a the symbol of the sequence $\{T_n\}$ and denote T_n by $T_n(a)$. The case where a is in $L^\infty(\mathbf{T})$ is easy, since then $\|T_n(a)\|\to\|a\|_\infty$ as $n\to\infty$. Things are more complicated for symbols a in $L^1(\mathbf{T})\setminus L^\infty(\mathbf{T})$. We here focus our attention on so-called Fisher-Hartwig symbols with a single singularity, that is, we consider functions a of the form

$$a(t)=|t-t_0|^{-2\alpha}\varphi_{\beta,t_0}(t)b(t)\quad(t\in\mathbf{T}),$$

where $t_0\in\mathbf{T}$, α is a complex number subject to the constraint $0<\operatorname{Re}\alpha<1/2$, β is a complex number satisfying $-1/2<\operatorname{Re}\beta\le 1/2$, the function φ_{β,t_0} is defined as

$$\varphi_{\beta,t_0}(t)=\exp(i\beta\arg(-t/t_0))\quad(t\in\mathbf{T})$$

with $\arg z\in(-\pi,\pi]$, and b is a function in $L^\infty(\mathbf{T})$ that is continuous at t_0 and does not vanish at t_0. The hypothesis $0<\operatorname{Re}\alpha<1/2$ ensures that $a\in L^1(\mathbf{T})\setminus L^\infty(\mathbf{T})$. We should mention that if a is any piecewise continuous function on $\mathbf{T}$ with a single jump, say at $t_0\in\mathbf{T}$, and $a(t_0\pm 0)\ne 0$, then a can be written in the form $a=\varphi_{\beta,t_0}b$ with $-1/2<\operatorname{Re}\beta\le 1/2$ and a continuous function b. Indeed, since $\varphi_{\beta,t_0}(t_0-0)=e^{\pi i\beta}$ and $\varphi_{\beta,t_0}(t_0+0)=e^{-\pi i\beta}$, it suffices to choose $\operatorname{Im}\beta\in(-\infty,\infty)$ and $\operatorname{Re}\beta\in(-1/2,1/2]$ so that

$$\frac{a(t_0+0)}{a(t_0-0)}=e^{-2\pi i\beta}=e^{2\pi\operatorname{Im}\beta}\,e^{-2\pi i\operatorname{Re}\beta}.$$

Our main result says that

$$\|T_n(a)\|\sim C_{\alpha,\beta}\,n^{2\operatorname{Re}\alpha}\,|b(t_0)|\quad\text{as}\quad n\to\infty$$

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where $C_{\alpha,\beta}$ is a completely identified constant depending only on α and β and where $x_n \sim y_n$ means that $x_n/y_n \to 1$. We will also establish results for symbols with more than one Fisher-Hartwig singularity.

For the exciting story behind Toeplitz matrices with Fisher-Hartwig symbols and their determinants we refer to the books [4], [5] and the papers [1], [9]. For general Toeplitz matrices, the asymptotic distribution of the singular values and the asymptotics of the extreme singular values has been studied by many authors, and we allow us to abstain from giving an ample list of references here. These investigations are mainly directed to the collective distribution of the singular values (Szegö-Avram-Parter theorems) or to the behavior of the extreme singular values of $T_n(a)$ for symbols a in $L^\infty(\mathbf{T})$. The asymptotics of the smallest singular value is governed by the nature of the zeros of the symbol a. This implies that the rate of convergence of the largest singular value (= the norm) of $T_n(a)$ to $\|a\|_\infty$ depends on the zeros of the function $\|a\|_\infty - |a(t)|$. These results are not applicable to Toeplitz matrices with symbols in $L^1(\mathbf{T}) \setminus L^\infty(\mathbf{T})$ or to Toeplitz matrices "without symbols." Such matrices are considered in [2], [3], [12], [13], [14], [15], for example, but the focus of these papers is not on the problem we are interested in here.

Under the sole assumption that b be in $L^\infty(\mathbf{T})$, the method of [2] yields the estimate $\|T_n(a)\| \le C_2\, n^{2\,\mathrm{Re}\,\alpha}$ with some finite constant C_2. If α is real, $\varphi_\beta b$ is real-valued and $\operatorname{essinf} b > 0$, one can also proceed as in [2] to show the existence of a positive constant C_1 such that $\|T_n(a)\| \ge C_1\, n^{2\,\mathrm{Re}\,\alpha}$. Such estimates were also derived in [11] by different arguments. These two-sided bounds are useful in several contexts (see [11], for example), but they are clearly far away from the precise asymptotics $\|T_n(a)\| \sim C_{\alpha,\beta}\, n^{2\,\mathrm{Re}\,\alpha}\, |b(t_0)|$.

The approach of the present paper is based on an idea of Harold Widom [16], [17], [18]: we construct integral operators K_n on $L^2(0,1)$ such that $\|T_n(a)\| = n^{2\,\mathrm{Re}\,\alpha}\, \|K_n\|$ and prove that K_n converges to some integral operator K in the operator norm on on $L^2(0,1)$, which implies that $\|K_n\| \to \|K\|$.

For nonnegative symbols, the results of this paper are of interest in the analysis of time series with long memory. The nth covariance matrix of a time series is a positive definite Toeplitz matrix $T_n(a) = (a_{j-k})_{j,k=1}^n$ and one wants to know its largest eigenvalue. If the series has a short memory, then a_n goes rapidly to zero as $|n| \to \infty$ and hence $\{a_n\}$ is the sequence of the Fourier coefficients of a function $a \in L^\infty(\mathbf{T})$. However, in the case of a long range memory, the numbers a_n may be of the order $|n|^{2\alpha-1}$ $(0 < \alpha < 1/2)$, which leads to symbols $a \in L^1(\mathbf{T}) \setminus L^\infty(\mathbf{T})$. The symbol $a(t) = |t - t_0|^{-2\,\alpha}\, b(t)$ is especially popular and will be considered in detail in Section 5. For more on Toeplitz matrices in time series we refer the reader to [6], [7], [8], [10], [11].

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2. A Special Class of Toeplitz Matrices. We begin with a simple observation.

Proposition 2.1. *Let γ be a real number. If $|a_{\pm n}| = O(n^\gamma)$ as $n \to \infty$, then $\|T_n\|$ converges to a finite limit for $\gamma < -1$, $\|T_n\| = O(\log n)$ for $\gamma = -1$, $\|T_n\| = O(n^{\gamma+1})$ for $\gamma > -1$. If $|a_{\pm n}| = o(n^\gamma)$ as $n \to \infty$, then $\|T_n\| = o(\log n)$ for $\gamma = -1$ and $\|T_n\| = o(n^{\gamma+1})$ for $\gamma > -1$.*

Proof. In the case $\gamma < -1$, the sequence $\{a_k\}$ is the sequence of the Fourier coefficients of a continuous function a and hence $\|T_n\| = \|T_n(a)\| \to \|a\|_\infty$. The spectral

norm of a Toeplitz matrix one diagonal of which is occupied by units and the remaining diagonals of which are zero equals 1. This implies that $\|T_n\| \le \sum_{k=-(n-1)}^{n-1} |a_k|$ and therefore yields the assertions concerning $\gamma = -1$ and $\gamma > -1$. $\square$

Let $A_n = (a_{j,k})_{j,k=0}^{n-1}$ be an $n \times n$ matrix with complex entries. We denote by G_n the integral operator on $L^2(0,1)$ with the kernel

$$g_n(x,y) = a_{[nx],[ny]}, \quad (x,y) \in (0,1)^2,$$

where $[\xi]$ denotes the integral part of ξ.

LEMMA 2.2. (WIDOM) *The spectral norm of A_n and the operator norm of G_n are related by the equality $\|A_n\| = n\|G_n\|$.*

Proof. Put $I_k = (k/n, (k+1)/n)$ and consider the operators

$$S_n : \{x_k\}_{k=0}^{n-1} \mapsto \sqrt{n} \sum_{k=0}^{n-1} x_k \chi_{I_k}, \quad T_n : f \mapsto \left\{ \sqrt{n} \int_{I_k} f(x)dx \right\}_{k=0}^{n-1}.$$

It is easily seen that $\|S_n\| = \|T_n\| = 1$ and that $T_n S_n$ is the identity operator on $\mathbf{C}^n$. Since $S_n A_n T_n = nG_n$ and thus $A_n = nT_n G_n S_n$ we obtain that $\|A_n\| \ge n\|G_n\|$ and $\|A_n\| \le n\|G_n\|$. $\square$

Let C^+, C^-, γ be complex numbers, let $\operatorname{Re}\gamma > -1$, let $a_{\pm n} = C^{\pm} n^{\gamma}$ for $n \ge 1$, and let a_0 be any complex number. Denote by K_n and K the integral operators on $L^2(0,1)$ with the kernels

$$k_n(x,y) = n^{-\gamma}\, a_{[nx]-[ny]} \quad \text{and} \quad k(x,y) = \begin{cases} C^+ (x-y)^{\gamma} & \text{for} \quad x > y, \\ C^-(y-x)^{\gamma} & \text{for} \quad x < y, \end{cases}$$

respectively.

LEMMA 2.3. *The operators K_n converge to K in the operator norm on $L^2(0,1)$.*

Proof. Fix a $\mu \in (0,1)$ sufficiently close to 1 such that $(1-\mu)|\operatorname{Re}\gamma| < \mu$ and $2\,\mu \operatorname{Re}\gamma < 1 + 2\operatorname{Re}\gamma$. Put

$$k_n^1(x,y) = \begin{cases} k(x,y) & \text{if } |x-y| > n^{\mu-1}, \\ 0 & \text{otherwise}, \end{cases} \quad k_n^2(x,y) = \begin{cases} k(x,y) & \text{if } |x-y| < n^{\mu-1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\ell_n^1(x,y) = \begin{cases} k_n(x,y) & \text{if } |x-y| > n^{\mu-1}, \\ 0 & \text{otherwise}, \end{cases} \quad \ell_n^2(x,y) = \begin{cases} k_n(x,y) & \text{if } |x-y| < n^{\mu-1}, \\ 0 & \text{otherwise}, \end{cases}$$

and denote by $K_n^1, K_n^2, L_n^1, L_n^2$ the integral operators on $L^2(0,1)$ with the kernels $k_n^1, k_n^2, \ell_n^1, \ell_n^2$, respectively. We have $K = K_n^1 + K_n^2$ and $K_n = L_n^1 + L_n^2$. Thus,

$$\|K - K_n\| \le \|K_n^1 - L_n^1\| + \|K_n^2\| + \|L_n^2\|.$$

We show that each term on the right goes to zero as $n \to \infty$.

To prove that $\|K^1 - K_n^1\| \to 0$ it suffices to show that $|k_n^1(x,y) - \ell_n^1(x,y)|$ converges uniformly to zero for $|x-y| > n^{\mu-1}$. We may assume that $x > y$, since the case $x < y$ can be tackled analogously. Thus, let $x - y > n^{\mu-1}$. As $[nx] - [ny] = n(x-y) + \varepsilon_n$ with $|\varepsilon_n| = |\varepsilon_n(x,y)| \le 2$, we get

$$\begin{aligned} \ell_n^1(x,y) &= C^+ n^{-\gamma}([nx]-[ny])^{\gamma} = C^+ n^{-\gamma}(n(x-y)+\varepsilon_n)^{\gamma} \\ &= C^+(x-y)^{\gamma} \left(1 + \frac{\varepsilon_n}{n(x-y)}\right)^{\gamma}. \end{aligned}$$

Since $n(x-y) > n^\mu$, it follows that $\ell_n^1(x,y) = C^+(x-y)^\gamma(1+O(n^{-\mu}))$ uniformly in x and y. Hence

$$|k_n^1(x,y) - \ell_n^1(x,y)| = |(x-y)^\gamma|\, O(n^{-\mu}) = (x-y)^{\operatorname{Re}\gamma}\, O(n^{-\mu})$$

uniformly in x and y. If $\operatorname{Re}\gamma \geq 0$, this goes to zero uniformly in x and y. In the case where $\operatorname{Re}\gamma < 0$, we use the inequality $x - y > n^{\mu-1}$ to obtain that

$$(x-y)^{\operatorname{Re}\gamma}\, O(n^{-\mu}) = O\left(n^{(1-\mu)|\operatorname{Re}\gamma|} n^{-\mu}\right)$$

uniformly in x and y, which is $o(1)$ because $(1-\mu)|\operatorname{Re}\gamma| < \mu$. We so have proved that $\|K^1 - K_n^1\| \to 0$ as $n \to \infty$.

The operator K_n^2 is the compression to $L^2(0,1)$ of the operator of convolution on $L^2(\mathbf{R})$ by the kernel

$$\kappa(x) = \begin{cases} C^+ x^\gamma & \text{for } \ 0 < x < n^{\mu-1}, \\ C^-|x|^\gamma & \text{for } \ -n^{\mu-1} < x < 0, \\ 0 & \text{for } \ |x| > n^{\mu-1}. \end{cases}$$

The norm of a convolution operator on $L^2(\mathbf{R})$ is the maximum of the modulus of the Fourier transform

$$(F\kappa)(\xi) = \int_{\mathbf{R}} \kappa(x) e^{i\xi x} dx \quad (\xi \in \mathbf{R})$$

of its convolution kernel $\kappa(x)$. Hence

$$\begin{aligned} \|K_n^2\| &\leq \max_{\xi\in\mathbf{R}} |(F\kappa)(\xi)| \leq \int_{\mathbf{R}} |\kappa(x)| dx \\ &= \int_{-n^{\mu-1}}^{1} C^-|x|^{\operatorname{Re}\gamma} dx + \int_0^{n^{\mu-1}} C^+ x^{\operatorname{Re}\gamma} dx = O\left(n^{(\mu-1)(\operatorname{Re}\gamma+1)}\right), \end{aligned}$$

which proves that $\|K_n^2\| \to 0$ as $n \to \infty$.

Let us consider the norm $\|L_n^2\|$. The kernel $\ell_n^2(x,y)$ is supported in the strip $|x-y| < n^{\mu-1}$. Let $\tilde{\ell}_n^2(x,y)$ be $k_n(x,y)$ for (x,y) in the staircase-like bordered strip $|[nx] - [ny]| < n^\mu$ and be zero otherwise. Denote by $\tilde{L}_n^2$ the corresponding integral operator. The difference $\ell_n^2(x,y) - \tilde{\ell}_n^2(x,y)$ is supported in about $4(n - n^\mu) = O(n)$ squares of side length $1/n$, and in these squares the absolute value of the difference is about $n^{-\gamma} a_{\pm[n^\mu]} = O(n^{-\operatorname{Re}\gamma} n^{\mu \operatorname{Re}\gamma})$. Consequently, the squared Hilbert-Schmidt norm $\|L_n^2 - \tilde{L}_n^2\|_2^2$ is at most a constant times $n\, n^{-2\operatorname{Re}\gamma} n^{2\,\mu \operatorname{Re}\gamma}(1/n)^2$, which goes to zero because $1 - 2\operatorname{Re}\gamma + 2\,\mu\operatorname{Re}\gamma - 2 = 2\,\mu\operatorname{Re}\gamma - (1 + 2\operatorname{Re}\gamma) < 0$. We are therefore left with proving that $\|\tilde{L}_n^2\| \to 0$. Let $T_n = (b_{j-k})_{j,k=0}^{n-1}$ where $b_k = a_k$ for $|k| \leq n^\mu$ and $b_k = 0$ otherwise. Lemma 2.2 implies that $\|\tilde{L}_n^2\| = (1/n) n^{-\operatorname{Re}\gamma} \|T_n\|$, and since

$$\|T_n\| \leq \sum_{k=-n^\mu}^{n^\mu} |b_k| = O\left(\sum_{k=-n^\mu}^{n^\mu} k^{\operatorname{Re}\gamma}\right) = O\left(n^{\mu(\operatorname{Re}\gamma+1)}\right),$$

we finally get $\|\tilde{L}_n^2\| = O\left(n^{(\mu-1)(\operatorname{Re}\gamma+1)}\right) = o(1)$. $\square$

THEOREM 2.4. *Let $T_n = (a_{j-k})_{j,k=0}^{n-1}$ where $a_{\pm n} = C^\pm n^\gamma(1 + o(1))$ as $n \to \infty$ with complex numbers C^+, C^-, γ such that $\operatorname{Re}\gamma > -1$ and at least one of the numbers C^+ and C^- is nonzero. Then*

$$\|T_n\| \sim \|K\|\, n^{\operatorname{Re}\gamma+1},$$

where K is the integral operator on $L^2(0,1)$ whose kernel is $C^+(x-y)^\gamma$ for $x>y$ and $C^-(y-x)^\gamma$ for $x<y$.

Proof. Write $T_n = S_n + D_n$ with $S_n = (b_{j-k})_{j,k=0}^{n-1}$, $D_n = (d_{j-k})_{j,k=0}^{n-1}$, $b_{\pm n} = C^\pm n^\gamma$, $d_{\pm n} = o(n^\gamma)$. From Lemma 2.2 we deduce that $\|S_n\|/n$ equals the norm of the integral operator $n^\gamma K_n$ where K_n has the kernel $n^{-\gamma} b_{[nx]-[ny]}$. Lemma 2.3 implies that $\|K_n - K\| \to 0$ and thus $\|K_n\| \to \|K\|$. Consequently,

$$\|S_n\| = \|K\| n^{\operatorname{Re}\gamma+1}(1+o(1)).$$

Proposition 2.1 yields $\|D_n\| = o(n^{\operatorname{Re}\gamma+1})$. □

THEOREM 2.5. *Let $B_1^+,\ldots,B_Q^+,B_1^-,\ldots,B_Q^-$ be complex numbers and suppose at least one of these numbers is nonzero. Let further $\gamma_1,\ldots,\gamma_Q$ be complex numbers such that $\operatorname{Re}\gamma_s > -1$ for all s. Put $\omega = e^{2\pi i/Q}$. Let $T_n = (a_{j-k})_{j,k=0}^{n-1}$ where*

$$a_{\pm n} = \sum_{s=1}^{Q} B_s^\pm \omega^{\pm sn} n^{\gamma_s}(1+o(1)) \quad \text{as } n\to\infty.$$

Put $\operatorname{Re}\gamma := \max_s \operatorname{Re}\gamma_s$ and let $S = \{s : \operatorname{Re}\gamma_s = \operatorname{Re}\gamma\}$. Then

$$\|T_n\| \sim \max_{s\in S} \|K_s\|\, n^{\operatorname{Re}\gamma+1},$$

where K_s is the integral operator on $L^2(0,1)$ whose kernel is $B_s^+(x-y)^{\gamma_s}$ for $x>y$ and $B_s^-(y-x)^{\gamma_s}$ for $x<y$.

Proof. Assume first that $n = mQ$ with a natural number m. We rearrange the rows of T_{mQ} by first taking the rows $1, Q+1, 2Q+1, \ldots$, then the rows $2, Q+2, 2Q+2, \ldots$, and so on. Then we make the same rearrangement with the columns. The resulting matrix has the same spectral norm as T_{mQ} and is a block Toeplitz matrix $(A_{j-k})_{j,k=0}^{Q-1}$ whose blocks are the Toeplitz matrices $A_k = (a_{k+(u-v)Q})_{u,v=0}^{m-1}$. For $0 \le |k| \le Q-1$, let D_k be the Toeplitz matrix $D_k = (d_{u-v}^{(k)})_{u,v=0}^{m-1}$ given by $d_0^{(k)} = 0$ and

$$d_{\pm\nu}^{(k)} = \sum_{s=1}^{Q} B_s^\pm \omega^{sk} (\nu Q)^{\gamma_s} \qquad (\nu \ge 1).$$

If $\nu\to\infty$, then eventually $k+\nu Q \ge 1$ and hence

$$a_{k+\nu Q} - d_\nu^{(k)} = \sum_{s=1}^{Q} B_s^+ \omega^{sk}\left[(k+\nu Q)^{\gamma_s}(1+o(1)) - (\nu Q)^{\gamma_s}\right].$$

The modulus of the term in brackets is

$$\begin{aligned}&(\nu Q)^{\operatorname{Re}\gamma_s}\left|\left(1+\frac{k}{\nu Q}\right)^{\gamma_s}(1+o(1)) - 1\right| \\ &= (\nu Q)^{\operatorname{Re}\gamma_s}|(1+o(1))(1+o(1)) - 1| = (\nu Q)^{\operatorname{Re}\gamma_s} o(1) = o(\nu^{\operatorname{Re}\gamma_s}).\end{aligned}$$

An analogous estimate holds for $\nu\to-\infty$. From Proposition 2.1 we therefore deduce that $A_k = D_k + E_k$ with $\|E_k\| = o(m^{\operatorname{Re}\gamma+1})$. It follows that

$$\|T_{mQ}\| = \|(D_{j-k})_{j,k=0}^{Q-1}\| + o(m^{\operatorname{Re}\gamma+1}).$$

Now, for $s \in \{1, \dots, Q\}$, put $H_s = (h_{u-v}^{(s)})_{u,v=0}^{m-1}$ where $h_0^{(s)} = 0$ and $h_{\pm\nu}^{(s)} = B_s^{\pm}(\nu Q)^{\gamma_s}$ for $\nu \ge 1$. Then

$$D_{j-k} = \sum_{s=1}^{Q} \omega^{s(j-k)} H_s.$$

Let $F_\pm$ be the block Fourier matrices $F_\pm = (\omega^{\pm jk} I_{m\times m})_{j,k=1}^{Q}$. The j,k block entry of $F_+ \operatorname{diag}(H_1, \dots, H_Q)\, F_-$ equals $\sum_{s=1}^{Q} \omega^{js} H_s \omega^{-sk} = D_{j-k}$. Consequently,

$$(D_{j-k})_{j,k=0}^{Q-1} = (D_{j-k})_{j,k=1}^{Q} = F_+ \operatorname{diag}(H_1, \dots, H_Q)\, F_-.$$

Since the matrices $(1/\sqrt{Q})\, F_\pm$ are unitary, we get

$$\|(D_{j-k})_{j,k=0}^{Q-1}\| = \sqrt{Q}\ \|\operatorname{diag}(H_1, \dots, H_Q)\|\ \sqrt{Q} = Q \max_s \|H_s\|.$$

Theorem 2.4 gives

$$\|H_s\| = m^{\operatorname{Re}\gamma_s}\, Q^{\operatorname{Re}\gamma_s + 1}\, \|K_s\|\,(1 + o(1)).$$

In summary,

$$\begin{aligned} \|T_{mQ}\| &= Q \max_s m^{\operatorname{Re}\gamma_s+1}\, Q^{\operatorname{Re}\gamma_s}\, \|K_s\|\,(1+o(1)) + o(m^{\operatorname{Re}\gamma+1}) \\ &= (mQ)^{\operatorname{Re}\gamma+1} \max_{s\in S} \|K_s\| + o(m^{\operatorname{Re}\gamma+1}) \\ &\sim (mQ)^{\operatorname{Re}\gamma+1} \max_{s\in S} \|K_s\|. \end{aligned}$$

Finally, if n is not divisible by Q, we can obtain T_n from T_{mQ} by adding at most $Q-1$ rows and columns. The spectral norm of a matrix with a single nonzero row or column is the ℓ^2 norm of this row or column, which in the case at hand does not exceed the square root of $\sum_{j=-(n-1)}^{n-1} |a_j|^2 = O(n^{2\operatorname{Re}\gamma+1})$, that is, $O(n^{\operatorname{Re}\gamma+1/2}) = o(n^{\operatorname{Re}\gamma+1})$. This completes the proof. $\square$

3. A Single Fisher-Hartwig Singularity. We first consider the pure Fisher-Hartwig singularity at $t_0 = 1$, that is, the function

$$\sigma(t) = |t-1|^{-2\alpha}\varphi_{\beta,1}(t)$$

with $0 < \operatorname{Re}\alpha < 1/2$ and $-1/2 < \operatorname{Re}\beta \le 1/2$. The Fourier coefficients of σ are

$$\sigma_n = (-1)^n \frac{\Gamma(1-2\alpha)}{\Gamma(-\alpha+\beta+1-n)\Gamma(-\alpha-\beta+1+n)},$$

with the convention that $\sigma_n := 0$ for $n < 0$ if $\alpha = -\beta$ and $\sigma_n := 0$ for $n > 0$ if $\alpha = \beta$ (see [5, Lemma 6.18]). Using the formula

$$\Gamma(1-z) = \frac{\pi z}{\sin \pi z}\, \frac{1}{\Gamma(1+z)}$$

we see that

$$\begin{aligned} \sigma_n &= (-1)^n\, \Gamma(1-2\alpha)\, \frac{\sin \pi(n+\alpha-\beta)}{\pi(n+\alpha-\beta)}\, \frac{\Gamma(n+1+\alpha-\beta)}{\Gamma(n+1-\alpha-\beta)} \\ &= \Gamma(1-2\alpha)\, \frac{\sin \pi(\alpha-\beta)}{\pi(n+\alpha-\beta)}\, \frac{\Gamma(n+1+\alpha-\beta)}{\Gamma(n+1-\alpha-\beta)} \end{aligned}$$

for $n \geq 0$ and

$$\begin{aligned}\sigma_{-n} &= (-1)^n \frac{\Gamma(1-2\alpha)}{\Gamma(-\alpha+\beta+1+n)\Gamma(-\alpha-\beta+1-n)} \\ &= (-1)^n \Gamma(1-2\alpha) \frac{\sin \pi(n+\alpha+\beta)}{\pi(n+\alpha+\beta)} \frac{\Gamma(n+1+\alpha+\beta)}{\Gamma(n+1-\alpha+\beta)} \\ &= \Gamma(1-2\alpha) \frac{\sin \pi(\alpha+\beta)}{\pi(n+\alpha+\beta)} \frac{\Gamma(n+1+\alpha+\beta)}{\Gamma(n+1-\alpha+\beta)}\end{aligned}$$

for $n \geq 0$. The asymptotic formula $\Gamma(n+\gamma)/\Gamma(n+\delta) \sim n^{\gamma-\delta}$ $(n \to \infty)$ shows that

$$\sigma_n = C^+_{\alpha,\beta}\, n^{2\alpha-1}(1+o(1)), \quad \sigma_{-n} = C^-_{\alpha,\beta}\, n^{2\alpha-1}(1+o(1))$$

as $n \to \infty$, where

$$C^{\pm}_{\alpha,\beta} = \Gamma(1-2\alpha) \frac{\sin \pi(\alpha \mp \beta)}{\pi}.$$

We denote by K the integral operator on $L^2(0,1)$ with the kernel

$$k(x,y) = \begin{cases} C^+_{\alpha,\beta}\,(x-y)^{2\alpha-1} & \text{for} \quad x > y, \\ C^-_{\alpha,\beta}\,(y-x)^{2\alpha-1} & \text{for} \quad x < y. \end{cases}$$

Obviously, $\|K\| > 0$.

THEOREM 3.1. *Suppose* $\sigma(t) = |t-t_0|^{-2\alpha}\varphi_{\beta,t_0}(t)$ *with* $t_0 \in \mathbf{T}$, $0 < \operatorname{Re}\alpha < 1/2$, $-1/2 < \operatorname{Re}\beta \leq 1/2$. *Then*

$$\|T_n(\sigma)\| \sim \|K\|\, n^{2\operatorname{Re}\alpha}.$$

Proof. We slightly change notation and denote the function σ defined as $\sigma(t) = |t-1|^{-2\alpha}\varphi_{\beta,1}(t)$ by σ^0. The σ of the present theorem results from σ^0 by replacing $t_0 = 1$ with a general $t_0 \in \mathbf{T}$. The only change in the Fourier coefficients is that the $(-1)^n$ in $(\sigma^0)_n$ becomes $(-1/t_0)^n$ in σ_n and hence $T_n(\sigma) = \Lambda\, T_n(\sigma^0)\, \Lambda^{-1}$ where $\Lambda := \operatorname{diag}(1, t_0^{-1}, \ldots, t_0^{-(n-1)})$. Therefore $\|T_n(\sigma)\| = \|T_n(\sigma^0)\|$. Taking into account that $(\sigma^0)_{\pm n} = C^{\pm}_{\alpha,\beta}\, n^{2\alpha-1}\,(1+o(1))$ and using Theorem 2.4 we arrive at the desired formula. $\square$

PROPOSITION 3.2. *If* σ *is as in Theorem 3.1 and* $c \in L^\infty(\mathbf{T})$ *is continuous and zero at* t_0, *then*

$$\|T_n(\sigma c)\| = o(n^{2\operatorname{Re}\alpha}).$$

Proof. Without loss of generality assume that $t_0 = 1$. Writing $\sigma = \operatorname{Re}\sigma + i\operatorname{Im}\sigma$ and $c = \operatorname{Re} c + i \operatorname{Im} c$ we get $T_n(\sigma c) = T_n(\operatorname{Re}\sigma \operatorname{Re} c) + \ldots$ (four terms) and thus $\|T_n(\sigma c)\| \leq \|T_n(\operatorname{Re}\sigma\operatorname{Re} c)\| + \ldots$. The matrix $T_n(\operatorname{Re}\sigma\operatorname{Re} c)$ is Hermitian and hence

$$\begin{aligned}\|T_n(\operatorname{Re}\sigma\operatorname{Re} c)\| &= \max_{\psi \in \mathbf{C}^n \setminus\{0\}} \frac{|(T_n(\operatorname{Re}\sigma\operatorname{Re} c)\psi, \psi)|}{\|\psi\|^2} \\ &= \max_{\varphi \in \mathcal{P}_n \setminus\{0\}} \frac{1}{\|\varphi\|^2} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re}\sigma(x) \operatorname{Re} c(x)\, |\varphi(x)|^2 dx \right| \\ &\leq \max_{\varphi \in \mathcal{P}_n \setminus\{0\}} \frac{1}{\|\varphi\|^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\operatorname{Re}\sigma(x)|\, |\operatorname{Re} c(x)|\, |\varphi(x)|^2 dx,\end{aligned}$$

where $\mathcal{P}_n$ is the set of all trigonometric polynomials of the form $\varphi(x) = \varphi_0 + \varphi_1 e^{ix} + \ldots + \varphi_{n-1} e^{i(n-1)x}$. Notice that

$$\begin{aligned}\|\varphi\|_\infty^2 &= \|\varphi_0 + \varphi_1 e^{ix} + \ldots + \varphi_{n-1} e^{i(n-1)x}\|_\infty^2 \le (|\varphi_0| + |\varphi_1| + \ldots + |\varphi_{n-1}|)^2 \\ &\le n\,(|\varphi_0|^2 + |\varphi_1|^2 + \ldots + |\varphi_{n-1}|^2) = \frac{n}{2\pi}\int_{-\pi}^{\pi} |\varphi(x)|^2 dx = \frac{n}{2\pi}\,\|\varphi\|^2.\end{aligned}$$

Clearly,

$$\operatorname{Re}\sigma(x) = \left|2\sin\frac{x}{2}\right|^{-2\operatorname{Re}\alpha} \cos\Big(\operatorname{Im}\alpha\,\log\left|2\sin\frac{x}{2}\right|\Big)\;|\varphi_{\beta,1}(x)|,$$

which is $O(|x|^{-2\operatorname{Re}\alpha})$ as $x \to 0$. We split the integral into $\int_{|x|<\pi/n}$, $\int_{\pi/n<|x|<\pi/\sqrt{n}}$, and $\int_{\pi/\sqrt{n}<|x|<\pi}$. The integral over $|x| < \pi/n$ is at most

$$\begin{aligned}&C_1 \sup_{|x|<\pi/n} |\operatorname{Re} c(x)|\,\|\varphi\|_\infty^2 \int_{|x|<\pi/n} |x|^{-2\operatorname{Re}\alpha} dx \\ &\le C_2 \sup_{|x|<\pi/n} |\operatorname{Re} c(x)|\,\frac{n}{2\pi}\,\|\varphi\|^2\, n^{2\operatorname{Re}\alpha-1} = o(n^{2\operatorname{Re}\alpha})\,\|\varphi\|^2\end{aligned}$$

because $\operatorname{Re} c(x) \to 0$ as $x \to 0$; here sup means esssup. The integral over the interval $\pi/n < |x| < \pi/\sqrt{n}$ has the upper bound

$$\begin{aligned}&C_3 \sup_{|x|<\pi/\sqrt{n}} |\operatorname{Re} c(x)| \int_{|x|>\pi/n} |x|^{-2\operatorname{Re}\alpha} |\varphi(x)|^2 dx \\ &\le C_3 \sup_{|x|<\pi/\sqrt{n}} |\operatorname{Re} c(x)|\,\frac{n^{2\operatorname{Re}\alpha}}{\pi^{2\operatorname{Re}\alpha}} \int_{|x|>\pi/n} |\varphi(x)|^2 dx \\ &\le C_3 \sup_{|x|<\pi/\sqrt{n}} |\operatorname{Re} c(x)|\,\frac{n^{2\operatorname{Re}\alpha}}{\pi^{2\operatorname{Re}\alpha}}\,\|\varphi\|^2 = o(n^{2\operatorname{Re}\alpha})\,\|\varphi\|^2,\end{aligned}$$

again because $\operatorname{Re} c(x) \to 0$ as $x \to 0$. Finally, the integral over $|x| > \pi/\sqrt{n}$ does not exceed

$$\begin{aligned}&C_4\,\|\operatorname{Re} c\|_\infty \int_{|x|>\pi/\sqrt{n}} |x|^{-2\operatorname{Re}\alpha} |\varphi(x)|^2 dx \\ &\le C_4\,\|\operatorname{Re} c\|_\infty\,\frac{n^{\operatorname{Re}\alpha}}{\pi^{2\operatorname{Re}\alpha}} \int_{|x|>\pi/\sqrt{n}} |\varphi(x)|^2 dx \\ &\le C_4\,\|\operatorname{Re} c\|_\infty\,\frac{n^{\operatorname{Re}\alpha}}{\pi^{2\operatorname{Re}\alpha}}\,\|\varphi\|^2 = O(n^{\operatorname{Re}\alpha})\,\|\varphi\|^2 = o(n^{2\operatorname{Re}\alpha})\,\|\varphi\|^2.\end{aligned}$$

This proves that $\|T_n(\operatorname{Re}\sigma\operatorname{Re} c)\| = o(n^{2\operatorname{Re}\alpha})$. Analogously one can show that

$$\|T_n(\operatorname{Re}\sigma\operatorname{Im} c)\|, \quad \|T_n(\operatorname{Im}\sigma\operatorname{Re} c)\|, \quad \|T_n(\operatorname{Im}\sigma\operatorname{Im} c)\|$$

are $o(n^{2\operatorname{Re}\alpha})$. □

THEOREM 3.3. *Let $a = \sigma b$ where σ is as in Theorem 3.1 and b is a function in $L^\infty(\mathbf{T})$ that is continuous at t_0 and does not vanish at t_0. Then*

$$\|T_n(a)\| \sim \|K\|\,|b(t_0)|\,n^{2\operatorname{Re}\alpha}.$$

Proof. We have $b(t) = b(t_0) + c(t)$ with $b(t_0) \neq 0$ and a function $c \in L^\infty(\mathbf{T})$ that is continuous and zero at t_0. It follows that $T_n(a) = b(t_0)\, T_n(\sigma) + T_n(\sigma c)$. Theorem 3.1 yields

$$\|b(t_0)\, T_n(\sigma)\| = |b(t_0)|\, \|T_n(\sigma)\| = |b(t_0)|\, \|K\|\, n^{2\,\mathrm{Re}\,\alpha}\,(1 + o(1)),$$

and Proposition 3.2 gives $\|T_n(\sigma c)\| = o(n^{2\,\mathrm{Re}\,\alpha})$. □

4. Several Fisher-Hartwig Singularities. Let $R \geq 2$ and

$$a(t) = b(t) \prod_{r=1}^{R} |t - t_r|^{-2\alpha_r} \varphi_{\beta_r, t_r}(t) \quad (t \in \mathbf{T})$$

where $t_1, \ldots, t_R$ are distinct points on $\mathbf{T}$, $0 < \mathrm{Re}\,\alpha_r < 1/2$, $-1/2 < \mathrm{Re}\,\beta_r \leq 1/2$, $b \in L^\infty(\mathbf{T})$, b is continuous at the points $t_1, \ldots, t_R$, and $b(t_r) \neq 0$ for all r. It is easily seen that a can be written in the form

$$a(t) = \sum_{r=1}^{R} |t - t_r|^{-2\alpha_r} \varphi_{\beta_r, t_r}(t)\, b_r(t) \quad (t \in \mathbf{T})$$

with functions $b_r \in L^\infty(\mathbf{T})$ such that b_r is continuous at t_r and satisfies $b_r(t_r) \neq 0$. Let

$$\mathrm{Re}\,\alpha := \max\{\mathrm{Re}\,\alpha_1, \ldots, \mathrm{Re}\,\alpha_R\}, \quad M = \{r : \mathrm{Re}\,\alpha_r = \mathrm{Re}\,\alpha\}.$$

If there is only one r_0 such that $\mathrm{Re}\,\alpha_{r_0} = \mathrm{Re}\,\alpha$, then Theorem 3.3 implies that

$$\|T_n(a)\| \sim \|K_{r_0}\|\, |b_{r_0}(t_{r_0})|\, n^{2\,\mathrm{Re}\,\alpha},$$

where K_r denotes the integral operator on $L^2(0,1)$ associated with $|t - t_r|^{-2\alpha_r} \varphi_{\beta_r, t_r}(t)$, that is, the integral operator whose kernel is $C^+_{\alpha_r, \beta_r}(x-y)^{2\,\alpha_r - 1}$ for $x > y$ and equals $C^-_{\alpha_r, \beta_r}(y-x)^{2\,\alpha_r - 1}$ for $x < y$. The case where the maximum is attained at more than one r is more difficult.

Conjecture 4.1. *We have*

$$\|T_n(a)\| \sim \max_{r \in M} \|K_r\|\, |b(t_r)|\, n^{2\,\mathrm{Re}\,\alpha}.$$

The following result confirms this conjecture in a sufficiently interesting special case.

Theorem 4.2. *If there is a $t_0 \in \mathbf{T}$ such that, for every r, $t_r = e^{2\pi i \varphi_r} t_0$ with a rational number φ_r, then Conjecture 4.1 is true.*

Proof. As passage from $a(t)$ to $a(t/t_0)$ does not change the spectral norm of the Toeplitz matrix (recall the proof of Theorem 3.1), we may without loss of generality assume that $t_0 = 1$. Put $\sigma_{\alpha,\beta,\tau}(t) = |t - \tau|^{-2\alpha} \varphi_{\beta,\tau}(t)$. The Fourier coefficients of a are

$$a_n = \sum_{r=1}^{R} (\sigma_{\alpha_r, \beta_r, t_r}\, b_r)_n = \sum_{r=1}^{R} b_r(t_r)\, (\sigma_{\alpha_r, \beta_r, t_r})_n + f_n$$

where $\{f_n\}$ is the sequence of the Fourier coefficients of a function $f \in L^1(\mathbf{T})$ for which $\|T_n(f)\| = o(n^{2\,\mathrm{Re}\,\alpha})$ (Proposition 3.2). Furthermore, $(\sigma_{\alpha_r,\beta_r,t_r})_n = t_r^{-n}(\sigma_{\alpha_r,\beta_r,1})_n$ (see once more the proof of Theorem 3.1). Thus,

$$a_n = \sum_{r=1}^{R} b_r(t_r)\, t_r^{-n}\,(\sigma_{\alpha_r,\beta_r,1})_n + f_n.$$

Let $t_r^{-1} = e^{2\pi i p_r/q_r}$ with a rational number $p_r/q_r \in (0,1]$ and denote by Q the least common multiple of $q_1,\dots,q_R$. Put $\omega = e^{2\pi i/Q}$. Then each t_r^{-1} is of the form ω^{k_r} with some $k_r \in \{1,2,\dots,Q\}$. It follows that

$$a_n = \sum_{r=1}^{R} b_r(t_r)\,\omega^{k_r n}\,(\sigma_{\alpha_r,\beta_r,1})_n + f_n$$

with different $k_1,\dots,k_R$ belonging to $\{1,2,\dots,Q\}$. From Section 3 we know that

$$(\sigma_{\alpha,\beta,1})_{\pm n} = C^{\pm}_{\alpha,\beta}\, n^{2\alpha-1}(1+o(1)).$$

Hence

$$a_{\pm n} = \sum_{r=1}^{R} b_r(t_r)\,\omega^{\pm k_r n}\, C^{\pm}_{\alpha_r,\beta_r}\, n^{2\alpha_r-1}(1+o(1)) + f_n,$$

which can be written as

$$a_{\pm n} = \sum_{s=1}^{Q} B_s^{\pm}\,\omega^{\pm sn}\, n^{\gamma_s}(1+o(1)) + f_n$$

with $B^{\pm}_{k_r} = b_r(t_r)\, C^{\pm}_{\alpha_r,\beta_r}$, $\gamma_{k_r} = 2\alpha_r - 1$ and $B_s^{\pm} = 0$, $\gamma_s = 0$ otherwise. Theorem 2.5 shows that the spectral norm of the Toeplitz matrix T_n^0 generated by

$$a^0_{\pm n} := \sum_{s=1}^{Q} B_s^{\pm}\,\omega^{\pm sn}\, n^{\gamma_s}(1+o(1))$$

satisfies

$$\|T_n^0\| \sim \max_{s\in S} \|K_s^0\|\, n^{2\,\mathrm{Re}\,\alpha} \quad \text{with} \quad \mathrm{Re}\,\alpha := \max_s \mathrm{Re}\,\alpha_s, \quad S = \{s : \mathrm{Re}\,\alpha_s = \mathrm{Re}\,\alpha\},$$

where K_s^0 is the operator whose kernel is $B_s^+(x-y)^{\gamma_s}$ for $x > y$ and $B_s^-(y-x)^{\gamma_s}$ for $x < y$. This is equivalent to saying that

$$\|T_n^0\| \sim \max_{r\in M} |b_r(t_r)|\,\|K_r\|\, n^{2\,\mathrm{Re}\,\alpha}$$

where the kernel of K_r is $C^+_{\alpha_r,\beta_r}(x-y)^{2\,\alpha_r-1}$ for $x > y$ and $C^-_{\alpha_r,\beta_r}(y-x)^{2\,\alpha_r-1}$ for $x < y$. Since $\|T_n(f)\| = o(n^{2\,\mathrm{Re}\,\alpha})$, we obtain that $\|T_n\| \sim \|T_n^0\|$. $\square$

5. A Particular Singularity. We finally embark on the case where

$$a(t) = |t - t_0|^{-2\alpha} b(t) \quad (t \in \mathbf{T})$$

with a real number $\alpha \in (0, 1/2)$ and a function $b \in L^\infty(\mathbf{T})$ that is continuous and nonzero at t_0. Theorem 3.3 gives

$$\|T_n(a)\| \sim \Gamma(1 - 2\alpha) \frac{\sin \pi\alpha}{\pi} \|K_\alpha\| \, |b(t_0)| \, n^{2\alpha}$$

where the kernel of K_α is $|x - y|^{2\alpha - 1}$.

PROPOSITION 5.1. *We have*

$$\frac{1}{2\alpha} \left(\frac{2}{4\alpha + 1} + 2 \frac{\Gamma(2\alpha + 1)\Gamma(2\alpha + 1)}{\Gamma(4\alpha + 2)} \right)^{1/2} \leq \|K_\alpha\| \leq \frac{1}{\alpha}.$$

Proof. We may think of K_α as the compression to $L^2(0, 1)$ of the convolution operator on $L^2(\mathbf{R})$ whose convolution kernel $\kappa(x)$ is $|x|^{2\alpha-1}$ for $|x| < 1$ and 0 for $|x| > 1$. As in the proof of Lemma 2.3 we therefore see that

$$\|K_\alpha\| \leq \max_{\xi \in \mathbf{R}} |(F\kappa)(\xi)| \leq \int_{\mathbf{R}} |\kappa(x)| \, dx = \int_{-1}^{1} |x|^{2\alpha - 1} \, dx = \frac{1}{\alpha}.$$

Let $\mathbf{1}$ be the function which is identically 1 on $(0, 1)$. Taking into account that

$$\|K_\alpha\|^2 \geq \|K_\alpha \mathbf{1}\|^2 / \|\mathbf{1}\|^2 = \|K_\alpha \mathbf{1}\|^2 \quad \text{and} \quad (K_\alpha \mathbf{1})(x) = \frac{1}{2\alpha} (x^{2\alpha} + (1 - x)^{2\alpha}),$$

we obtain that $\|K_\alpha\|^2$ is greater than or equal to

$$\frac{1}{4\alpha^2} \int_0^1 (x^{2\alpha} + (1 - x)^{2\alpha})^2 \, dx = \frac{1}{4\alpha^2} \left(\frac{2}{4\alpha + 1} + 2 \frac{\Gamma(2\alpha + 1)\Gamma(2\alpha + 1)}{\Gamma(4\alpha + 2)} \right).$$

This proves the lower bound for $\|K_\alpha\|$. □

COROLLARY 5.2. *We have* $\|K_\alpha\| \sim 1/\alpha$ *as* $\alpha \to 0$ *and* $\|K_\alpha\| \sim 1$ *as* $\alpha \to 1/2$.

Proof. By Proposition 5.1, $\alpha^2 \|K_\alpha\|^2 \leq 1$ and

$$\liminf_{\alpha \to 0} \alpha^2 \|K_\alpha\|^2 \geq \frac{1}{4} \left(2 + 2 \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} \right) = 1,$$

which implies that $\alpha \|K_\alpha\| \to 1$ as $\alpha \to 0$. Thinking of $K_\alpha - K_{1/2}$ as the convolution operator with the convolution kernel $|x|^{2\alpha - 1} - 1$ for $|x| < 1$ and 0 for $|x| > 1$, we get

$$\|K_\alpha - K_{1/2}\| \leq \int_{-1}^{1} (|x|^{2\alpha - 1} - 1) \, dx = \frac{1}{\alpha} - 2 = o(1) \text{ as } \alpha \to \frac{1}{2}.$$

Thus, $\|K_\alpha\| \to \|K_{1/2}\|$ as $\alpha \to 1/2$. Since $(K_{1/2} f)(x) = \int_0^1 f(y) \, dy$, it is easily seen that $\|K_{1/2}\| = 1$. □

COROLLARY 5.3. *We have*

$$\Gamma(1 - 2\alpha) \frac{\sin \pi\alpha}{\pi} \|K_\alpha\| \sim 1 \text{ as } \alpha \to 0,$$

$$\Gamma(1 - 2\alpha) \frac{\sin \pi\alpha}{\pi} \|K_\alpha\| \sim \frac{1}{2\pi(1/2 - \alpha)} \text{ as } \alpha \to \frac{1}{2}.$$

Proof. The asymptotics for $\alpha \to 0$ is immediate from Corollary 5.2. For $\alpha \to 1/2$, Corollary 5.2 and the formulas

$$\Gamma(1-2\alpha)\,\frac{\sin \pi\alpha}{\pi} \sim \frac{\Gamma(1-2\alpha)}{\pi} = \frac{1}{\sin 2\pi\alpha}\,\frac{1}{\Gamma(2\alpha)} \sim \frac{1}{2\pi(1/2-\alpha)},$$

yield the asserted asymptotics. □

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